

# A Diffusion Synthetic Acceleration Method for the $S_N$ Equations With Discontinuous Finite Element Space and Time Differencing

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## Abstract

A diffusion synthetic acceleration method is developed for the time dependent  $S_N$  equations with linear discontinuous finite element time differencing and discontinuous finite element spatial differencing on unstructured grids. Both theoretical and computational results are given which demonstrate the effectiveness and efficiency of the method.

## 1 Introduction

Various numerical schemes have been developed and used for the time-dependent  $S_N$  equations. Many of these schemes such as step, diamond and weighted diamond, involve only the zero time moment of the angular flux. Other advanced differencing schemes, such as the finite-moments methods [Badruzzaman, 1991] and discontinuous finite-element methods (DFEM), involve higher time moments of the angular flux. When standard Source Iteration (SI) is used as the solution method, it is often desirable or even necessary to use diffusion synthetic acceleration (DSA) [Alcouffe, 1977] to accelerate both the zero and higher time moments of the scattering source. For advanced time differencing schemes the standard four step DSA technique [Larsen, 1982] leads to coupled DSA equations for the zero and higher time moments of the scalar flux corrections that are extremely difficult to solve efficiently. This difficulty of the four-step DSA equations is further compounded when advanced spatial differencing schemes are used. In fact, even the time-independent four-step DSA equations are difficult to solve efficiently with advanced spatial schemes. In this paper we develop a new DSA method for solving the  $S_N$  equations with linear DFEM (LDFEM) time differencing and DFEM spatial differencing on unstructured grids. This new DSA method results in de-coupled equations that can be efficiently solved with standard iterative solution techniques.

The paper will proceed as follows: in Section 2, we describe the analytic problem to be solved; in Section 3 we describe the new DSA method for the LDFEM time differenced  $S_N$  equations with no spatial differencing; in Section 4 we extend the new DSA method to include DFEM spatial differencing on unstructured grids; in Section 5, we present timing and iteration count results from an unstructured tetrahedral mesh problem; and in Section 6 we finish with some concluding remarks.

## 2 Analytic Time-Dependent DSA

The problem to be solved is the single group, time-dependent  $S_N$  equations in Cartesian coordinates with isotropic scattering:

$$\hat{\Omega}_m \cdot \vec{\nabla} \psi_m(\vec{r}, t) + \frac{1}{v} \frac{\partial}{\partial t} \psi_m(\vec{r}, t) + \sigma_t(\vec{r}) \psi_m(\vec{r}, t) = \sigma_s(\vec{r}) \phi(\vec{r}, t) + q(\vec{r}, t), \quad (1)$$

$$\phi(\vec{r}, t) = \sum_{m=1}^M \psi_m(\vec{r}, t) w_m.$$

with appropriate boundary and initial conditions. The SI and DSA equations are given by the following:

$$\hat{\Omega}_m \cdot \vec{\nabla} \psi_m^{(\ell+1/2)} + \frac{1}{v} \frac{\partial}{\partial t} \psi_m^{(\ell+1/2)} + \sigma_t \psi_m^{(\ell+1/2)} = \sigma_s \phi^{(\ell)} + q, \quad (2)$$

$$\vec{\nabla} \cdot \vec{\delta J}^{(\ell+1)} + \frac{1}{v} \frac{\partial}{\partial t} \delta \phi^{(\ell+1)} + \sigma_a \delta \phi^{(\ell+1)} = \sigma_s \left( \phi^{(\ell+1/2)} - \phi^{(\ell)} \right), \quad (3)$$

$$\frac{1}{3} \vec{\nabla} \delta \phi^{(\ell+1)} + \frac{1}{v} \frac{\partial}{\partial t} \vec{\delta J}^{(\ell+1)} + \sigma_t \vec{\delta J}^{(\ell+1)} = 0, \quad (4)$$

$$\phi^{(\ell+1)} = \phi^{(\ell+1/2)} + \delta \phi^{(\ell+1)}. \quad (5)$$

We note that Eqs.(3) and (4) are obtained by taking the angular  $P_1$  approximation to Eq.(2). In the following sections, we consider LDFEM time differencing and then DFEM spatial differencing of the above equations.

## 3 LDFEM Time Differenced DSA Method

In this section, we develop a new DSA method for the LDFEM time-differenced  $S_N$  equations in Cartesian geometry with no spatial differencing. We first introduce a time mesh with subscript “ $n$ ” referring to the  $n$ -th time element. The upper and lower boundaries of the time element are  $t_{n+1/2}$  and  $t_{n-1/2}$  and the time element width is given by  $\Delta t_n = t_{n+1/2} - t_{n-1/2}$ . For clarity we drop the “ $m$ ” subscripts. The LDFEM time differencing of Eq.(2) is given by:

$$\hat{\Omega} \cdot \vec{\nabla} \psi_n^{(\ell+1/2)} + \frac{1}{v \Delta t_n} \left( \psi_{n+1/2}^{(\ell+1/2)} - \psi_{n-1/2}^{(\ell+1/2)} \right) + \sigma_t \psi_n^{(\ell+1/2)} = \sigma_s \phi_n^{(\ell)} + q_n, \quad (6)$$

$$\hat{\Omega} \cdot \vec{\nabla} \psi_n^{t,(\ell+1/2)} + \frac{3}{v \Delta t_n} \left( \psi_{n+1/2}^{(\ell+1/2)} + \psi_{n-1/2}^{(\ell+1/2)} - 2\psi_n^{(\ell+1/2)} \right) + \sigma_t \psi_n^{t,(\ell+1/2)} = \sigma_s \phi_n^{t,(\ell)} + q_n^t, \quad (7)$$

$$\psi_{n+1/2}^{(\ell+1/2)} = \psi_n^{(\ell+1/2)} + \psi_n^{t,(\ell+1/2)}, \quad (8)$$

where  $\psi_n^{(\ell+1/2)}$ ,  $\phi_n^{(\ell)}$ , and  $q_n$  refer to the average angular flux, scalar flux and fixed source in time, respectively, and  $\psi_n^{t,(\ell+1/2)}$ ,  $\phi_n^{t,(\ell)}$ , and  $q_n^t$  refer to the first time moment of the angular flux, scalar flux and fixed source, respectively. Equations (6) and (7) can be rewritten in terms of  $\psi_n$  and  $\psi_n^t$  exclusively:

$$\begin{aligned}\hat{\Omega} \cdot \vec{\nabla} \psi_n^{(\ell+1/2)} + \frac{1}{v\Delta t_n} \psi_n^{t,(\ell+1/2)} + \left( \sigma_t + \frac{1}{v\Delta t_n} \right) \psi_n^{(\ell+1/2)} \\ = \sigma_s \phi_n^{(\ell)} + q_n + \frac{1}{v\Delta t_n} (\psi_{n-1} + \psi_{n-1}^t),\end{aligned}\quad (9)$$

$$\begin{aligned}\hat{\Omega} \cdot \vec{\nabla} \psi_n^{t,(\ell+1/2)} - \frac{3}{v\Delta t_n} \psi_n^{(\ell+1/2)} + \left( \sigma_t + \frac{3}{v\Delta t_n} \right) \psi_n^{t,(\ell+1/2)} \\ = \sigma_s \phi_n^{t,(\ell)} + q_n^t - \frac{3}{v\Delta t_n} (\psi_{n-1} + \psi_{n-1}^t).\end{aligned}\quad (10)$$

The LDFEM time differenced form of Eqs.(3) and (4) is given by

$$\vec{\nabla} \cdot \vec{\delta J}_n^{(l+1)} + \frac{1}{v\Delta t_n} \delta \phi_n^{t,(l+1)} + \left( \sigma_a + \frac{1}{v\Delta t_n} \right) \delta \phi_n^{(l+1)} = \sigma_s (\phi_n^{(\ell+1/2)} - \phi_n^{(\ell)}), \quad (11)$$

$$\frac{1}{3} \vec{\nabla} \delta \phi_n^{(l+1)} + \frac{1}{v\Delta t_n} \vec{\delta J}_n^{t,(l+1)} + \left( \sigma_t + \frac{1}{v\Delta t_n} \right) \vec{\delta J}_n^{(l+1)} = 0, \quad (12)$$

$$\vec{\nabla} \cdot \vec{\delta J}_n^{t,(l+1)} - \frac{3}{v\Delta t_n} \delta \phi_n^{(l+1)} + \left( \sigma_a + \frac{3}{v\Delta t_n} \right) \delta \phi_n^{t,(l+1)} = \sigma_s (\phi_n^{t,(\ell+1/2)} - \phi_n^{t,(\ell)}), \quad (13)$$

$$\frac{1}{3} \vec{\nabla} \delta \phi_n^{t,(l+1)} - \frac{3}{v\Delta t_n} \vec{\delta J}_n^{(l+1)} + \left( \sigma_t + \frac{3}{v\Delta t_n} \right) \vec{\delta J}_n^{t,(l+1)} = 0. \quad (14)$$

These equations are referred to as the standard DSA equations, which are a difficult set of coupled equations that cannot easily be solved nor reduced to a form that is easily solved. To obtain DSA equations that are easier to solve, we systematically simplify these equations. First, we replace Eqs.(12) and (14) with the diffusion approximation (Fick's Law):

$$\frac{1}{3} \vec{\nabla} \delta \phi_n^{(l+1)} + \sigma_t \vec{\delta J}_n^{(l+1)} = 0, \quad (15)$$

$$\frac{1}{3} \vec{\nabla} \delta \phi_n^{t,(l+1)} + \sigma_t \vec{\delta J}_n^{t,(l+1)} = 0. \quad (16)$$

We then de-couple Eqs.(11) and (13) by assuming the diamond approximation for  $\delta \phi_n^{t,(l+1)}$  in Eq.(11). The diamond approximation is given by:

$$\delta \phi_n^{(l+1)} = \frac{1}{2} (\delta \phi_{n+1/2}^{(l+1)} + \delta \phi_{n-1/2}^{(l+1)}),$$

and

$$\delta \phi_n^{t,(l+1)} = \frac{1}{2} (\delta \phi_{n+1/2}^{(l+1)} - \delta \phi_{n-1/2}^{(l+1)}).$$

Since,  $\delta \phi_{n-1/2} = 0$ , the diamond approximation gives

$$\delta \phi_n^{t,(l+1)} = \delta \phi_n^{(l+1)}. \quad (17)$$

Inserting Eqs.(17) and (15) into Eq.(11) and using Eq.(16) into Eq.(13), the simplified DSA accelerated transport iterations become

$$\begin{aligned}\hat{\Omega} \cdot \vec{\nabla} \psi_n^{(\ell+1/2)} + \frac{1}{v\Delta t_n} \psi_n^{t,(\ell+1/2)} + \left( \sigma_t + \frac{1}{v\Delta t_n} \right) \psi_n^{(\ell+1/2)} \\ = \sigma_s \phi_n^{(\ell)} + q_n + \frac{1}{v\Delta t_n} (\psi_{n-1} + \psi_{n-1}^t),\end{aligned}\quad (18)$$

$$\begin{aligned}\hat{\Omega} \cdot \vec{\nabla} \psi_n^{t,(\ell+1/2)} - \frac{3}{v\Delta t_n} \psi_n^{(\ell+1/2)} + \left( \sigma_t + \frac{3}{v\Delta t_n} \right) \psi_n^{t,(\ell+1/2)} \\ = \sigma_s \phi_n^{t,(\ell)} + q_n^t - \frac{3}{v\Delta t_n} (\psi_{n-1} + \psi_{n-1}^t),\end{aligned}\quad (19)$$

$$-\vec{\nabla} \cdot \frac{1}{3\sigma_t} \vec{\nabla} \delta \phi_n^{(l+1)} + \left( \sigma_a + \frac{2}{v\Delta t_n} \right) \delta \phi_n^{(l+1)} = \sigma_s (\phi_n^{(\ell+1/2)} - \phi_n^{(\ell)}), \quad (20)$$

$$-\vec{\nabla} \cdot \frac{1}{3\sigma_t} \vec{\nabla} \delta \phi_n^{t,(l+1)} + \left( \sigma_a + \frac{3}{v\Delta t_n} \right) \delta \phi_n^{t,(l+1)} = \sigma_s (\phi_n^{t,(\ell+1/2)} - \phi_n^{t,(\ell)}) + \frac{3}{v\Delta t_n} \delta \phi_n^{(l+1)}, \quad (21)$$

$$\phi_n^{(\ell+1)} = \phi_n^{(\ell+1/2)} + \delta \phi_n^{(l+1)}, \quad (22)$$

$$\phi_n^{t,(\ell+1)} = \phi_n^{t,(\ell+1/2)} + \delta \phi_n^{t,(l+1)}. \quad (23)$$

Note that Eq.(20) is de-coupled from Eq.(21) with Eq.(20) being solved first to obtain the average scalar flux correction. The results are then used in Eq.(21), which is solved for the first time moment of the scalar flux correction.

In order to justify our simplifications, we have performed a Fourier analysis on the simplified DSA accelerated transport iterations in both slab and x-y geometries. In this analysis, the scattering ratio is equal to unity and we use  $S_4$  quadrature. The slab geometry Fourier analysis spectral radii for both SI and DSA are given in Table 1 and the x-y geometry Fourier analysis spectral radii for both SI and DSA are given in Table 2. We see that the SI spectral radii approach unity as  $v\Delta t$  becomes large and as  $\sigma_t$  becomes large. The DSA method is unconditionally stable and effective, with the maximum spectral radii bounded considerably away from unity.

Table 1: SI and DSA Fourier Analysis Spectral Radii For Slab Geometry With No Spatial Differencing.

	$\sigma_t = 1$		$\sigma_t = 10$		$\sigma_t = 100$	
$v\Delta t$	SI	DSA	SI	DSA	SI	DSA
0.1	0.04	0.02	0.30	0.10	0.83	0.07
1.0	0.30	0.13	0.83	0.08	0.98	0.01
10.0	0.83	0.20	0.98	0.12	0.998	0.08
100.0	0.98	0.19	0.998	0.10	0.9998	0.10
1000.0	0.998	0.19	0.9998	0.10	0.99998	0.10

#### 4 DFEM Space Differencing and LDFEM Time Differenced DSA Method

In this section we consider the DFEM spatial differencing of Eqs.(18)-(23). We begin by assuming that the problem domain has been divided into a unstructured spatial grid of volume elements (spatial

Table 2: SI and DSA Fourier Analysis Spectral Radii For X-Y Geometry With No Spatial Differencing.

	$\sigma_t = 1$		$\sigma_t = 10$		$\sigma_t = 100$	
$v\Delta t$	SI	DSA	SI	DSA	SI	DSA
0.1	0.04	0.02	0.30	0.10	0.83	0.07
1.0	0.30	0.13	0.83	0.11	0.98	0.01
10.0	0.83	0.23	0.98	0.13	0.998	0.10
100.0	0.98	0.25	0.998	0.10	0.9998	0.10
1000.0	0.998	0.25	0.9998	0.10	0.99998	0.10

cells). The elements shapes can be tetrahedra, hexahedra, prisms, pyramids, etc. The material properties within each element are assumed to be constant. The DFEM formulation and implementation for time-independent problems has been described in [Wareing, et al., 1999].

The DFEM approximation for the  $k$ -th element is given by

$$\psi_n(\mathbf{r}) \approx \vec{\Theta}^T \vec{\psi}_n, \quad (24)$$

$$\psi_n^t(\mathbf{r}) \approx \psi_n^t, \quad (25)$$

where  $\vec{\Theta}^T$  is the transpose of a column vector of spatial interpolation (basis) functions,  $\vec{\psi}_n$  is a column vector of nodal angular fluxes, and the first time slope of the angular flux,  $\psi_n^t$ , is constant within the element. The DFEM spatial differencing of Eq.(18) is given by

$$\begin{aligned} & \int_{\delta V_k} \hat{\Omega}_i \hat{n}_i \vec{\Theta} \vec{\Theta}^T \vec{\psi}_n^{s,(\ell+1/2)} d\delta V - \int_{V_k} \frac{\partial \vec{\Theta}}{\partial r_i} \vec{\Theta}^T \vec{\psi}_n^{(\ell+1/2)} dV \\ & + \frac{\psi_n^{t,(\ell+1/2)}}{v\Delta t_n} \int_{V_k} \vec{\Theta} dV + \left( \sigma_t + \frac{1}{v\Delta t_n} \right) \int_{V_k} \vec{\Theta} \vec{\Theta}^T \vec{\psi}_n^{(\ell+1/2)} dV \\ & = \sigma_s \int_{V_k} \vec{\Theta} \vec{\Theta}^T \vec{\phi}_n^{(\ell)} dV + \int_{V_k} \vec{\Theta} q_n dV + \frac{1}{v\Delta t_n} \int_{V_k} \vec{\Theta} \vec{\Theta}_{n-1}^T \vec{\psi}_n dV + \frac{\psi_{n-1}^t}{v\Delta t_n} \int_{V_k} \vec{\Theta} dV, \end{aligned} \quad (26)$$

and the DFEM spatial differencing of Eq.(19) is given by

$$\begin{aligned} & \int_{\delta V_k} \hat{\Omega}_i \hat{n}_i \psi_n^{t,s,(\ell+1/2)} d\delta V - \frac{3}{v\Delta t_n} \int_{V_k} \vec{\Theta}^T \vec{\psi}_n^{(\ell+1/2)} dV + \left( \sigma_t + \frac{3}{v\Delta t_n} \right) V_k \psi_n^{t,(\ell+1/2)} \\ & = \sigma_s \phi_n^{(\ell)} V_k + \int_{V_k} q_n^t dV - \frac{3}{v\Delta t_n} \int_{V_k} \vec{\Theta}^T \vec{\psi}_{n-1} dV - \frac{3\psi_{n-1}^t}{v\Delta t_n} V_k. \end{aligned} \quad (27)$$

Here, we have used standard summation convention (i.e. a repeated index in the same multiplicative term implies a summation) for the  $\vec{\nabla}$  operator, the direction  $\hat{\Omega}$ , and the outward directed unit normal vector  $\hat{n}$ . The surface angular fluxes and first time moment of the angular flux are discontinuous and are given defined by:

$$\vec{\psi}_n^{s,(\ell+1/2)} = \begin{cases} \vec{\psi}_n^{(\ell+1/2)}, & \hat{\Omega}_i \hat{n}_i > 0 \\ \vec{\psi}_n^{inc,(\ell+1/2)}, & \hat{\Omega}_i \hat{n}_i < 0 \end{cases}, \quad (28)$$

$$\psi_n^{t,s,(\ell+1/2)} = \begin{cases} \psi_n^{t,(\ell+1/2)}, & \hat{\Omega}_i \hat{n}_i > 0 \\ \psi_n^{t,inc,(\ell+1/2)}, & \hat{\Omega}_i \hat{n}_i < 0 \end{cases}, \quad (29)$$

where,  $\vec{\psi}^{inc,(\ell+1/2)}$  is the corresponding column vector of nodal angular flux values of the element that shares the surface of the  $k$ -th element and  $\psi_n^{t,inc,(\ell+1/2)}$  is the first time moment of the angular flux of the element that shares the surface of the  $k$ -th element. We note that the surfaces of the elements are given by a finite set of lower dimensional elements. For example, with tetrahedral elements, the surface is comprised of four triangular elements

We now consider the DSA equations. To obtain a discretization of Eq.(20) one could certainly use the method of Adams and Martin [Adams and Martin, 1992]. However, efficient solvers have not yet been developed to solve these DFEM DSA equations on unstructured grids. At the present time, we use an adaptation [Wareing, et al., 1999] of the Wareing, Larsen and Adams method [Wareing, et al., 1991], which is an approximation to the Adams and Martin method. Here, the DFEM DSA equations are replaced by continuous finite element (CFEM) DSA equations plus a local within-element mapping procedure to project from the CFEM scalar flux corrections,  $\delta\phi_{n,cont}^{(\ell+1)}$ , to the approximated DFEM scalar flux corrections,  $\delta\phi_n^{(\ell+1)}$ . The CFEM equations are derived by considering the contribution from element  $k$  to the individual vertices forming element  $k$  given by

$$\begin{aligned} & -\frac{1}{3\sigma_t} \int_{\delta V_k} n_i \vec{\Theta} \frac{\partial \vec{\Theta}^T}{\partial r_i} \delta\phi_{n,cont}^{(\ell+1)} dV + \frac{1}{3\sigma_t} \int_{V_k} \left( \frac{\partial \vec{\Theta}}{\partial r_i} \right) \left( \frac{\partial \vec{\Theta}^T}{\partial r_j} \right) \delta\phi_{n,cont}^{(\ell+1)} dV \\ & + \left( \sigma_a + \frac{2}{v\Delta t_n} \right) \int_{V_k} \vec{\Theta} \vec{\Theta}^T \delta\phi_{n,cont}^{(\ell+1)} dV = \sigma_s \int_{V_k} \vec{\Theta} \vec{\Theta}^T \left( \vec{\phi}_n^{(\ell+1/2)} - \vec{\phi}_n^{(\ell)} \right) dV. \end{aligned} \quad (30)$$

A global CFEM matrix is formed for all vertices in the mesh by summing the individual element contributions using the standard finite-element technique [Zienkiewicz, 1994]. This CFEM matrix is a  $N_{vertex} \times N_{vertex}$  symmetric positive-definite matrix, where  $N_{vertex}$  is the number of vertices in the mesh. Marshak boundary conditions are used for all boundary vertices. The local within-element mapping from continuous scalar flux corrections to discontinuous scalar flux corrections is given by the following:

$$\begin{aligned} & \int_{\delta V_k} \vec{\Theta} \vec{\Theta}^T \frac{\delta\phi_n^{(\ell+1)}}{2} dV + \left( \sigma_a + \frac{2}{v\Delta t_n} \right) \int_{V_k} \vec{\Theta} \vec{\Theta}^T \delta\phi_n^{(\ell+1)} dV \\ & = \int_{\delta V_k} \vec{\Theta} \vec{\Theta}^T \frac{\delta\phi_{n,cont}^{(\ell+1)}}{2} dV \sigma_s \int_{V_k} \vec{\Theta} \vec{\Theta}^T \left( \vec{\phi}_n^{(\ell+1/2)} - \vec{\phi}_n^{(\ell)} \right) dV. \end{aligned} \quad (31)$$

The accelerated scalar fluxes are given by

$$\vec{\phi}_n^{(\ell+1)} = \vec{\phi}_n^{(\ell+1/2)} + \delta\phi_n^{(\ell+1)}. \quad (32)$$

We note that this simplified DSA method applied to time-independent problems is conditionally effective and the effectiveness can degrade with skewed and high aspect ratio elements.

The discretization of Eq.(21) for the first time moment of the scalar flux correction requires considerable more detail. We begin by applying the DFEM approximation which gives

$$\begin{aligned} & -\frac{1}{3\sigma_t} \int_{\delta V_k} \hat{n}_i \frac{\partial}{\partial r_i} \delta\phi_n^{t,s,(\ell+1)} dV + \left( \sigma_t + \frac{3}{v\Delta t_n} \right) V_k \delta\phi_n^{t,(\ell+1)} \\ & = \sigma_s V_k \left( \phi_n^{t,(\ell+1/2)} - \phi_n^{t,(\ell)} \right) + \frac{3}{v\Delta t_n} \int_{V_k} \vec{\Theta}^T \delta\phi_n^{(\ell+1)} dV. \end{aligned} \quad (33)$$

We have labeled the surface first time moment of the scalar flux correction as  $\delta\phi_n^{t,s,(\ell+1)}$ , which is discontinuous. We define this surface quantity using the usual partial-current approach:

$$-\frac{\hat{n}_i}{3\sigma_t} \frac{\partial}{\partial r_i} \delta\phi_n^{t,s,(\ell+1)} = \frac{\delta\phi_n^{t,(\ell+1)}}{4} - \frac{\delta\phi_n^{t,inc,(\ell+1)}}{4} - \frac{1}{2} \left( \frac{\hat{n}_i}{3\sigma_t} \frac{\partial}{\partial r_i} \delta\phi_n^{t,(\ell+1)} + \frac{\hat{n}_i}{3\sigma_t^i} \frac{\partial}{\partial r_i} \delta\phi_n^{t,inc,(\ell+1)} \right). \quad (34)$$

The problem with Eq.(34) is that  $\delta\phi_n^{t,(\ell+1)}$  and  $\delta\phi_n^{t,inc,(\ell+1)}$  are constant within their respective elements. Thus their gradients would appear to be zero. However, this assumption does not result in an effective acceleration scheme. Therefore we must use some type of generalized gradient expression in Eq.(34). We have chosen to use the following definition for the gradient of a function averaged over a volume:

$$\langle \vec{\nabla} \phi \rangle = \frac{1}{V} \int_{\delta V} \phi d\delta V, \quad (35)$$

Then for each face of the element, given the index,  $l$ , we define

$$\frac{1}{2} \left( \frac{\hat{n}_i}{3} \frac{\partial}{\partial r_i} \delta\phi_n^{t,(\ell+1)} + \frac{\hat{n}_i}{3} \frac{\partial}{\partial r_i} \delta\phi_n^{t,inc,(\ell+1)} \right) = \frac{2\delta V_{k,l} (\delta\phi_n^{t,inc,(\ell+1)} - \delta\phi_n^{t,(\ell+1)})}{3V_k + 3V_k^{inc}} \quad (36)$$

where the gradient has been averaged over both element volumes  $k$  and  $k'$ , where  $k'$  represents the index of the element that shares the  $l$ -th face of the  $k$ -th element. We heuristically extend this ‘‘averaged-gradient’’ concept to obtain the final expression we desire

$$\frac{1}{2} \left( \frac{\hat{n}_i}{3\sigma_t} \frac{\partial}{\partial r_i} \delta\phi_n^{t,(\ell+1)} + \frac{\hat{n}_i}{3\sigma_t^i} \frac{\partial}{\partial r_i} \delta\phi_n^{t,inc,(\ell+1)} \right) = \frac{2\delta V_{k,l} (\delta\phi_n^{t,inc,(\ell+1)} - \delta\phi_n^{t,(\ell+1)})}{3\sigma_{t,k} V_k + 3\sigma_{t,k'}^{inc} V_{k'}^{inc}}. \quad (37)$$

The discretization of Eq.(21) is then given by

$$\begin{aligned} & \sum_{l=1}^{N_{faces}} \left[ \frac{\delta V_{k,l}}{4} (\delta\phi_n^{t,(\ell+1)} - \delta\phi_n^{t,inc,(\ell+1)}) - \frac{2\delta V_{k,l} (\delta\phi_n^{t,inc,(\ell+1)} - \delta\phi_n^{t,(\ell+1)})}{3\sigma_{t,k} V_k + 3\sigma_{t,k'}^{inc} V_{k'}^{inc}} \right] \\ & + \left( \sigma_t + \frac{3}{v\Delta t_n} \right) V_k \delta\phi_n^{t,(\ell+1)} = \sigma_s V_k (\phi_n^{t,(\ell+1/2)} - \phi_n^{t,(\ell)}) + \frac{3}{v\Delta t_n} \int_{V_k} \vec{\Theta}^T \vec{\delta\phi}_n^{t,(\ell+1)} dV, \end{aligned} \quad (38)$$

with Marshak boundary conditions, where  $N_{faces}$  is the number of faces of the element. This equation leads to a  $N_{element} \times N_{element}$  symmetric positive definite matrix. Finally,

$$\phi_n^{t,(\ell+1)} = \phi_n^{t,(\ell+1/2)} + \delta\phi_n^{t,(\ell+1)}. \quad (39)$$

We note that our DSA method involves the solution of two de-coupled symmetric positive-definite matrices, which can be efficiently solved using standard solution techniques.

We have performed a Fourier analysis on the simplified DSA accelerated transport iterations in slab geometry with linear DFEM spatial differencing and in x-y geometry with bilinear DFEM spatial differencing. In this analysis, the scattering ratio is equal to unity and we use  $S_4$  quadrature. The slab and x-y geometry Fourier analysis predicted spectral radii for SI and DSA are given in Table 3 and Table 4, respectively, for various combinations of  $v\Delta t$ ,  $\sigma_t$ ,  $\Delta x$  and  $\Delta y$ . The trends in the spectral radii are the same as that for no spatial differencing. We see that the method is very effective most of the time. However, we do see a degradation in the spectral radii as the aspect ratio of the elements become large. This is a direct result of using the adaptation of the Wareing, Larsen and Adams DSA method for Eq.(20). The use of the Adams and Martin DSA method for Eq.(20) would eliminate this aspect ratio problem. From the Fourier analysis, we have found that the spectral radii for time-dependence is never larger than that for time-independent problems. We note that these Fourier analysis results have been computationally verified using a research and development transport code.

Table 3: SI and DSA Fourier Analysis Spectral Radii For Slab Geometry With LDFEM Spatial Differencing.

$v\Delta t$	$\Delta x$	$\sigma_t = 1$		$\sigma_t = 10$		$\sigma_t = 100$	
		SI	DSA	SI	DSA	SI	DSA
0.1	0.1	0.07	0.04	0.43	0.19	0.91	0.41
0.1	1.0	0.09	0.05	0.50	0.25	0.91	0.45
0.1	10.0	0.09	0.05	0.50	0.25	0.91	0.45
1.0	0.1	0.30	0.11	0.83	0.23	0.98	0.29
1.0	1.0	0.43	0.19	0.90	0.41	0.99	0.49
1.0	10.0	0.50	0.25	0.91	0.45	0.99	0.49
10.0	0.1	0.83	0.19	0.98	0.30	0.998	0.26
10.0	1.0	0.83	0.23	0.98	0.29	0.999	0.46
10.0	10.0	0.90	0.41	0.99	0.49	0.999	0.49

## 5 Unstructured Mesh Test Problem

This test problem is a sphere with a diameter of  $2.0\text{ cm}$  containing a  $1.0\text{ cm} \times 1.0\text{ cm} \times 1.0\text{ cm}$  cube in the center. Both the sphere and box are purely scattering media with a scattering ratio of unity. The total cross section in the sphere not containing the box is  $10\text{ cm}^{-1}$  and that in the box,  $\sigma_{t,box}$ , is to  $0.01\text{ cm}^{-1}$ ,  $1.0\text{ cm}^{-1}$  or  $10.0\text{ cm}^{-1}$ . There is a homogeneous source of strength  $1\frac{\text{particle}}{\text{cm}^3\text{-s}}$ . We set  $v\Delta t$  to  $0.1\text{ cm}$ ,  $1.0\text{ cm}$  or  $10.0\text{ cm}$ . The problem is meshed with unstructured tetrahedral elements (1735 elements total).  $S_4$  level-symmetric quadrature is used and the relative pointwise convergence criterion is  $10^{-4}$  for the scalar fluxes.

Table 5 provides the total CPU time and number of iterations for each configuration of  $v\Delta t$  and  $\sigma_{t,box}$  for both SI and the new DSA method. We see that the DSA method is both effective and efficient for all  $v\Delta t$  and  $\sigma_{t,box}$ . We note that the CPU time required to solve the two diffusion equations increases the CPU time per iteration by only  $\sim 5\%$  and thus, allows for unconditional efficiency for this problem. The efficiency will certainly increase as the quadrature order is increased.

## 6 Conclusions

We have successfully developed a DSA method for the  $S_N$  equations with LDFEM time differencing and DFEM spatial differencing on unstructured grids. The method is effective and should be efficient for most problems. The only limiting problem is the conditions under which the approximate DSA acceleration equations for the zero time moments of the scattering source is effective. We are currently investigating efficient solution techniques for unconditionally effective DFEM DSA equations. The acceleration equations for the first time moment of the scattering source do not present any additional problems to those encountered in time-independent problems.

Table 4: SI and DSA Fourier Analysis Spectral Radii For X-Y Geometry With LDFEM Spatial Differencing.

$v\Delta t$	$\Delta x$	$\Delta y$	$\sigma_t = 1$		$\sigma_t = 10$		$\sigma_t = 100$	
			SI	DSA	SI	DSA	SI	DSA
0.1	0.1	0.1	0.04	0.03	0.31	0.17	0.86	0.44
0.1	1.0	1.0	0.08	0.04	0.47	0.25	0.90	0.48
0.1	10.0	10.0	0.09	0.05	0.50	0.25	0.91	0.46
0.1	0.1	1.0	0.08	0.05	0.50	0.30	0.91	0.63
0.1	0.1	10.0	0.09	0.06	0.50	0.33	0.91	0.65
1.0	0.1	0.1	0.27	0.11	0.82	0.25	0.98	0.49
1.0	1.0	1.0	0.31	0.15	0.88	0.44	0.99	0.63
1.0	10.0	10.0	0.50	0.25	0.91	0.48	0.99	0.58
1.0	0.1	1.0	0.34	0.20	0.86	0.66	0.99	0.83
1.0	0.1	10.0	0.49	0.43	0.91	0.82	0.99	0.90
10.0	0.1	0.1	0.82	0.21	0.98	0.45	0.998	0.49
10.0	1.0	1.0	0.82	0.23	0.98	0.49	0.998	0.70
10.0	10.0	10.0	0.88	0.44	0.99	0.63	0.999	0.66
10.0	0.1	1.0	0.82	0.25	0.98	0.76	0.99	0.88
10.0	0.1	10.0	0.88	0.79	0.99	0.96	0.999	0.98

Table 5: CPU Time and Iteration Counts for the Unstructured Mesh Test Problem.

$v\Delta t$	$\sigma_{t,box}$	SI		DSA	
		CPU Time (s)	Iterations	CPU Time (s)	Iterations
0.1	0.01	57.1	19	47.8	14
0.1	0.1	57.1	19	47.3	14
0.1	1.0	59.9	20	46.7	14
0.1	10.0	62.6	21	49.2	15
1.0	0.01	140.3	50	63.3	19
1.0	0.1	140.3	50	59.6	18
1.0	1.0	203.2	66	52.6	16
1.0	10.0	219.4	79	60.9	19
10.0	0.01	330.2	116	51.2	15
10.0	0.1	328.8	115	47.6	14
10.0	1.0	333.6	117	44.1	13
10.0	10.0	440.0	154	43.7	13

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